

SYMMETRIC BILINEAR FORM ON A LIE ALGEBRA

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ABSTRACT. Let \mathfrak{g} be the finite dimensional simple Lie algebra associated to an indecomposable and symmetrizable generalized Cartan matrix $C = (a_{ij})_{n \times n}$ of finite type and let \mathfrak{d} be a finite dimensional Lie algebra related to a quantum group $D_{q,p^{-1}}(\mathfrak{g})$ obtained by Hodges, Levasseur and Toro [1] by deforming the quantum group $U_q(\mathfrak{g})$. Here we see that \mathfrak{d} is a generalization of \mathfrak{g} and give a \mathfrak{d} -invariant symmetric bilinear form on \mathfrak{d} .

Let $C = (a_{ij})_{n \times n}$ be an indecomposable and symmetrizable generalized Cartan matrix of finite type and let \mathfrak{g} be the finite dimensional simple Lie algebra associated to C . (Refer to [2, Chapter 2] and [4, Chapter 1, 2, 4] for details.) Hodges, Levasseur and Toro constructed a quantum group $D_{q,p^{-1}}(\mathfrak{g})$ in [1, Theorem 3.5] by deforming the quantum group $U_q(\mathfrak{g})$, which is considered as a generalization of $U_q(\mathfrak{g})$, and obtained a Hopf dual $\mathbb{C}_{q,p}[G]$ of $D_{q,p^{-1}}(\mathfrak{g})$ in [1, §3] that is a generalization of the Hopf algebra $\mathbb{C}_q[G]$ studied in [3] and [7, §3]. The second author constructed a finite dimensional Lie algebra \mathfrak{d} in [6, Theorem 1.3] by using a skew symmetric bilinear form u on a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . He showed in [6, Proposition 1.4] that \mathfrak{d} is a generalized Lie bialgebra of the standard Lie bialgebra given in \mathfrak{g} . Moreover he studied in [6, §3 and §5] the Poisson structure of the Hopf dual $\mathbb{C}[G]$ of the universal enveloping algebra $U(\mathfrak{d})$ that is considered as a Poisson version of $\mathbb{C}_{q,p}[G]$. In this note we see that \mathfrak{d} is a generalization of \mathfrak{g} (Proposition 3) and find a \mathfrak{d} -invariant symmetric bilinear form on \mathfrak{d} (Theorem 4).

We begin with explaining the notations in [6, 1.1]. Let $C = (a_{ij})_{n \times n}$ be an indecomposable and symmetrizable generalized Cartan matrix of finite type. Hence there exists a diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$, where all d_i are positive integers, such that the matrix DC is symmetric positive definite. (Each d_i is denoted by s_i and ϵ_i^{-1} in [2, §2.3] and [4, Chapter 2] respectively.) Throughout the paper, we denote by

$\mathfrak{g} = (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ the finite dimensional simple Lie algebra over the complex number field \mathbb{C} associated to C ,
 \mathfrak{h} a Cartan subalgebra of \mathfrak{g} with simple roots $\alpha_1, \dots, \alpha_n$.

Choose $h_i \in \mathfrak{h}$, $1 \leq i \leq n$, such that

$$(1) \quad \alpha_j : \mathfrak{h} \longrightarrow \mathbb{C}, \quad \alpha_j(h_i) = a_{ij} \quad \text{for all } j = 1, \dots, n.$$

2010 *Mathematics Subject Classification.* 17B67.

Key words and phrases. Finite dimensional Lie algebra, Symmetric bilinear form.

This research has been performed as a subproject of project Research for Applications of Mathematical Principles (No. C21501) and supported by the National Institute of Mathematics Science.

Then $\{h_i\}_{i=1}^n$ forms a basis of \mathfrak{h} , since C has rank n , and \mathfrak{g} is generated by h_i and $x_{\pm\alpha_i}$, $i = 1, \dots, n$, with relations

$$\begin{aligned} [h_i, h_j]_{\mathfrak{g}} &= 0, \quad [h_i, x_{\pm\alpha_j}]_{\mathfrak{g}} = \pm a_{ij} x_{\pm\alpha_j}, \quad [x_{\alpha_i}, x_{-\alpha_j}]_{\mathfrak{g}} = \delta_{ij} h_i, \\ (\text{ad}_{x_{\pm\alpha_i}})^{1-a_{ij}}(x_{\pm\alpha_j}) &= 0, \quad i \neq j \end{aligned}$$

by [2, Definition 2.1.3]. (In [2, Definition 2.1.3], x_{α_i} and $x_{-\alpha_i}$ are denoted by e_i and f_i respectively.) Denote by

- \mathfrak{n}^+ the subspace of \mathfrak{g} spanned by root vectors with positive roots
- \mathfrak{n}^- the subspace of \mathfrak{g} spanned by root vectors with negative roots

and set

$$\mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{n}^+.$$

Hence

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Henceforth we mean by x_{α} that x_{α} is a root vector of \mathfrak{n} with root α .

By [2, §2.3], there exists a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ on \mathfrak{h}^* given by

$$(2) \quad (\alpha_i|\alpha_j) = d_i a_{ij}$$

for all $i, j = 1, \dots, n$. This form $(\cdot|\cdot)$ induces the isomorphism

$$\mathfrak{h}^* \longrightarrow \mathfrak{h}, \quad \lambda \mapsto h_{\lambda},$$

where h_{λ} is defined by

$$(\alpha_i|\lambda) = \alpha_i(h_{\lambda}) \quad \text{for all } i = 1, \dots, n.$$

(The isomorphism $\mathfrak{h}^* \longrightarrow \mathfrak{h}, \lambda \mapsto h_{\lambda}$, is denoted by ν^{-1} in [2, §2.3] and [4, Chapter 2].) Identifying \mathfrak{h}^* to \mathfrak{h} via $\lambda \mapsto h_{\lambda}$, \mathfrak{h} has a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ given by

$$(\lambda|\mu) = (h_{\lambda}|h_{\mu}) = \lambda(h_{\mu}).$$

This is extended to a nondegenerate \mathfrak{g} -invariant symmetric bilinear form on \mathfrak{g} by [4, Theorem 2.2 and its proof] and [2, (2.7) and Proposition 2.3.6]:

$$(3) \quad (h_i|h_j) = d_j^{-1} a_{ij}, \quad (h|x_{\alpha}) = 0, \quad (x_{\alpha}|x_{\beta}) = 0 \text{ if } \alpha + \beta \neq 0, \quad (x_{\alpha_i}|x_{-\alpha_j}) = d_i^{-1} \delta_{ij}$$

for $h \in \mathfrak{h}$, root vectors x_{α}, x_{β} and $i = 1, \dots, n$.

Lemma 1. [6, Lemma 1.1] *For each positive root α ,*

$$(4) \quad [x_{\alpha}, x_{-\alpha}]_{\mathfrak{g}} = (x_{\alpha}|x_{-\alpha}) h_{\alpha}.$$

Let $u = (u_{ij})$ be a skew symmetric $n \times n$ -matrix with entries in \mathbb{C} . Then u induces a skew symmetric bilinear (alternating) form u on \mathfrak{h}^* given by

$$u(\lambda, \mu) := \sum_{i,j} u_{ij} \lambda(h_i) \mu(h_j)$$

for any $\lambda, \mu \in \mathfrak{h}^*$. Hence there exists a unique linear map $\Phi : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ such that

$$u(\lambda, \mu) = (\Phi(\lambda)|\mu)$$

for any $\lambda, \mu \in \mathfrak{h}^*$ since the form $(\cdot|\cdot)$ on \mathfrak{h}^* is nondegenerate. Set

$$\Phi_+ = \Phi + I, \quad \Phi_- = \Phi - I,$$

where I is the identity map on \mathfrak{h}^* . Thus

$$(5) \quad \begin{aligned} (\Phi + \lambda|\mu) &= u(\lambda, \mu) + (\lambda|\mu) \\ (\Phi - \lambda|\mu) &= u(\lambda, \mu) - (\lambda|\mu) \end{aligned}$$

for all $\lambda, \mu \in \mathfrak{h}^*$.

Fix a vector space \mathfrak{k} isomorphic to \mathfrak{h} and let

$$(6) \quad \varphi : \mathfrak{h} \longrightarrow \mathfrak{k}$$

be an isomorphism of vector spaces. For each $\lambda \in \mathfrak{h}^*$, denote by $k_\lambda \in \mathfrak{k}$ the element $\varphi(h_\lambda)$. Let

$$\mathfrak{g}' := \mathfrak{k} \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{k} \oplus \mathfrak{n}^+$$

be the Lie algebra isomorphic to \mathfrak{g} such that each element $k_\lambda \in \mathfrak{k}$ corresponds to the element h_λ and each root vector x_α corresponds to x_α . That is, \mathfrak{g}' is the Lie algebra with Lie bracket

$$(7) \quad \begin{aligned} [k_\lambda, k_\mu]_{\mathfrak{g}'} &= 0, & [k_\lambda, x_\alpha]_{\mathfrak{g}'} &= (\alpha|\lambda)x_\alpha, \\ [x_\alpha, x_\beta]_{\mathfrak{g}'} &= [x_\alpha, x_\beta]_{\mathfrak{g}} \quad (\alpha \neq -\beta), & [x_\alpha, x_{-\alpha}]_{\mathfrak{g}'} &= \varphi([x_\alpha, x_{-\alpha}]_{\mathfrak{g}}), \end{aligned}$$

where $\lambda, \mu \in \mathfrak{h}^*$ and $x_\alpha, x_{-\alpha}, x_\beta$ are root vectors with roots $\alpha, -\alpha, \beta$ respectively.

Definition 2. [6, Theorem 1.3] The vector space $\mathfrak{d} := \mathfrak{n}^- \oplus \mathfrak{k} \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a Lie algebra with Lie bracket

$$(8) \quad [h_\lambda, h_\mu] = 0, \quad [h_\lambda, k_\mu] = 0, \quad [k_\lambda, k_\mu] = 0,$$

$$(9) \quad [h_\lambda, x_\alpha] = -(\Phi - \lambda|\alpha)x_\alpha,$$

$$(10) \quad [k_\lambda, x_\alpha] = (\Phi + \lambda|\alpha)x_\alpha,$$

$$(11) \quad [x_\alpha, x_\beta] = 2^{-1}([x_\alpha, x_\beta]_{\mathfrak{g}} + [x_\alpha, x_\beta]_{\mathfrak{g}'})$$

for all $h_\lambda, h_\mu \in \mathfrak{h}$, $k_\lambda, k_\mu \in \mathfrak{k}$ and root vectors $x_\alpha, x_\beta \in \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{n}^+$.

The Lie algebra \mathfrak{d} is a generalization of \mathfrak{g} as seen in the following proposition.

Proposition 3. (a) Let \mathfrak{l} be the subspace of \mathfrak{d} spanned by all $h_\lambda - k_\lambda, \lambda \in \text{rad}(u)$, where

$$\text{rad}(u) = \{\lambda \in \mathfrak{h}^* | u(\lambda, \mathfrak{h}^*) = 0\}.$$

Then \mathfrak{l} is a solvable ideal of \mathfrak{d} . In particular, if $u = 0$ then \mathfrak{l} is the maximal solvable ideal of \mathfrak{d} and $\mathfrak{d}/\mathfrak{l}$ is isomorphic to \mathfrak{g} .

(b) Let \mathfrak{m} be the subspace of \mathfrak{d} spanned by all root vectors x_α and all $h_\lambda + k_\lambda, \lambda \in \mathfrak{h}^*$. Then \mathfrak{m} is an ideal of \mathfrak{d} isomorphic to \mathfrak{g} .

Proof. (a) Note that

$$[h_\lambda - k_\lambda, x_\alpha] = -2u(\lambda, \alpha)x_\alpha$$

for any root vector x_α . Hence if $\lambda \in \text{rad}(u)$ then $[h_\lambda - k_\lambda, x_\alpha] = 0$ for all root vectors x_α . It follows that \mathfrak{l} is an ideal of \mathfrak{d} . Clearly \mathfrak{l} is solvable by (8).

Suppose that $u = 0$. Then $\text{rad}(u) = \mathfrak{h}^*$ and thus the ideal \mathfrak{l} is the subspace spanned by all $h_\lambda - k_\lambda, \lambda \in \mathfrak{h}^*$, which is solvable. It is easy to see that $\mathfrak{g} \cong \mathfrak{d}/\mathfrak{l}$ by (11) and (7). Hence $\mathfrak{d}/\mathfrak{l}$ is simple and thus \mathfrak{l} is the unique maximal solvable ideal of \mathfrak{d} .

(b) Note that

$$[h_\lambda + k_\lambda, x_\alpha] = 2(\lambda|\alpha)x_\alpha$$

for any root vector x_α and

$$[x_\alpha, x_{-\alpha}] = 2^{-1}(x_\alpha | x_{-\alpha})(h_\alpha + k_\alpha)$$

for any positive root α by Lemma 1 and (11). Hence \mathfrak{m} is an ideal of \mathfrak{d} by (8)-(11). Moreover \mathfrak{m} is isomorphic to \mathfrak{g} since the linear map from \mathfrak{m} into \mathfrak{g} defined by

$$h_\lambda + k_\lambda \mapsto 2h_\lambda, \quad x_\alpha \mapsto x_\alpha \quad (\lambda \in \mathfrak{h}^*)$$

is a Lie algebra isomorphism. \square

We give a \mathfrak{d} -invariant symmetric bilinear form on \mathfrak{d} as in the following theorem.

Theorem 4. *Define a bilinear form $(\cdot | \cdot)_\mathfrak{d}$ on \mathfrak{d} by*

$$(12) \quad (h | h')_\mathfrak{d} = 2(h | h'), \quad (h | x)_\mathfrak{d} = (h | x) = 0,$$

$$(13) \quad (x | h)_\mathfrak{d} = (x | h) = 0, \quad (x | x')_\mathfrak{d} = (x | x'),$$

$$(14) \quad (h_\lambda | k_\mu)_\mathfrak{d} = -2u(\lambda, \mu), \quad (k_\mu | h_\lambda)_\mathfrak{d} = 2u(\mu, \lambda),$$

$$(15) \quad (k | x)_\mathfrak{d} = (\varphi^{-1}(k) | x) = 0, \quad (x | k)_\mathfrak{d} = (x | \varphi^{-1}(k)) = 0,$$

$$(16) \quad (k | k')_\mathfrak{d} = 2(\varphi^{-1}(k) | \varphi^{-1}(k'))$$

for $h, h', h_\lambda \in \mathfrak{h}$, $x, x' \in \mathfrak{n}^+ \oplus \mathfrak{n}^-$, $k, k', k_\mu \in \mathfrak{k}$, where φ is the isomorphism given in (6). Then $(\cdot | \cdot)_\mathfrak{d}$ is a \mathfrak{d} -invariant symmetric bilinear form.

Proof. Since $(\cdot | \cdot)$ is a symmetric bilinear form on \mathfrak{g} and u is a skew symmetric bilinear form on \mathfrak{h}^* , $(\cdot | \cdot)_\mathfrak{d}$ is clearly symmetric by (12)-(16).

Let us show that $(\cdot | \cdot)_\mathfrak{d}$ is \mathfrak{d} -invariant, that is,

$$(a | [b, c])_\mathfrak{d} = ([a, b] | c)_\mathfrak{d}$$

for all $a, b, c \in \mathfrak{d}$.

Case I. $a = x_\alpha, b = x_\beta, c = x_\gamma$:

(a) $\alpha + \beta + \gamma \neq 0$: Note that $(x_{\alpha'} | x_{\beta'})_\mathfrak{d} = (x_{\alpha'} | x_{\beta'}) = 0$ for root vectors $x_{\alpha'}, x_{\beta'}$ of \mathfrak{n} with $\alpha' + \beta' \neq 0$ by (3) and (13). Hence

$$([x_\alpha, x_\beta] | x_\gamma)_\mathfrak{d} = 0$$

since $[x_\alpha, x_\beta] \in \mathfrak{n}$ if $\alpha + \beta \neq 0$ and $[x_\alpha, x_\beta] \in \mathfrak{h} \oplus \mathfrak{k}$ if $\alpha + \beta = 0$ by (12) and (15). Similarly

$$(x_\alpha | [x_\beta, x_\gamma])_\mathfrak{d} = 0$$

since $[x_\beta, x_\gamma] \in \mathfrak{n}$ if $\beta + \gamma \neq 0$ and $[x_\beta, x_\gamma] \in \mathfrak{h} \oplus \mathfrak{k}$ if $\beta + \gamma = 0$ by (13) and (15). Hence

$$([x_\alpha, x_\beta] | x_\gamma)_\mathfrak{d} = 0 = (x_\alpha | [x_\beta, x_\gamma])_\mathfrak{d}.$$

(b) $\alpha + \beta + \gamma = 0$: Since $\alpha + \beta = -\gamma$, we have $[x_\alpha, x_\beta] = [x_\alpha, x_\beta]_\mathfrak{g}$ by (7) and (11). Similarly, since $\beta + \gamma = -\alpha$, we also have $[x_\beta, x_\gamma] = [x_\beta, x_\gamma]_\mathfrak{g}$ by (7) and (11). Hence we have that

$$([x_\alpha, x_\beta] | x_\gamma)_\mathfrak{d} = ([x_\alpha, x_\beta]_\mathfrak{g} | x_\gamma) = (x_\alpha | [x_\beta, x_\gamma]_\mathfrak{g}) = (x_\alpha | [x_\beta, x_\gamma])_\mathfrak{d}$$

by (13).

Case II. $a = h_\lambda \in \mathfrak{h}$, $b = x_\beta$, $c = x_\gamma$:

(a) $\beta + \gamma = 0$: Note that $x_\gamma = x_{-\beta}$. Since $[h_\lambda, x_\beta] = -(\Phi - \lambda | \beta)x_\beta$ by (9) and

$$[x_\beta, x_{-\beta}] = 2^{-1}([x_\beta, x_{-\beta}]_\mathfrak{g} + [x_\beta, x_{-\beta}]_{\mathfrak{g}'}) = 2^{-1}(x_\beta | x_{-\beta})h_\beta + 2^{-1}(x_\beta | x_{-\beta})k_\beta$$

by (11) and (4), we have that

$$\begin{aligned}
([h_\lambda, x_\beta]|x_{-\beta})_{\mathfrak{d}} &= -(\Phi - \lambda|\beta)(x_\beta|x_{-\beta})_{\mathfrak{d}} && \text{(by (9))} \\
&= (\lambda|\beta)(x_\beta|x_{-\beta}) - u(\lambda, \beta)(x_\beta|x_{-\beta}) && \text{(by (5), (13))} \\
&= 2^{-1}(h_\lambda|(x_\beta|x_{-\beta})h_\beta)_{\mathfrak{d}} + 2^{-1}(h_\lambda|(x_\beta|x_{-\beta})k_\beta)_{\mathfrak{d}} && \text{(by (12), (14))} \\
&= 2^{-1}(h_\lambda|[x_\beta, x_{-\beta}]_{\mathfrak{g}} + [x_\beta, x_{-\beta}]_{\mathfrak{g}'})_{\mathfrak{d}} && \text{(by (4))} \\
&= (h_\lambda|[x_\beta, x_{-\beta}])_{\mathfrak{d}}. && \text{(by (11))}
\end{aligned}$$

(b) $\beta + \gamma \neq 0$: Since $[x_\beta, x_\gamma] \in \mathfrak{n}$, we have that

$$([h_\lambda, x_\beta]|x_\gamma)_{\mathfrak{d}} = -(\Phi - \lambda|\beta)(x_\beta|x_\gamma)_{\mathfrak{d}} = 0 = (h_\lambda|[x_\beta, x_\gamma])_{\mathfrak{d}}$$

by (3), (9), (12) and (13).

Case III. $a = k_\lambda \in \mathfrak{k}$, $b = x_\beta$, $c = x_\gamma$: This case is to be shown as in Case II. We repeat it for completion.

(a) $\beta + \gamma = 0$: Note that $x_\gamma = x_{-\beta}$. Since

$$[x_\beta, x_{-\beta}] = 2^{-1}([x_\beta, x_{-\beta}]_{\mathfrak{g}} + [x_\beta, x_{-\beta}]_{\mathfrak{g}'}) = 2^{-1}(x_\beta|x_{-\beta})h_\beta + 2^{-1}(x_\beta|x_{-\beta})k_\beta$$

by (11) and (4), we have that

$$\begin{aligned}
([k_\lambda, x_\beta]|x_{-\beta})_{\mathfrak{d}} &= (\Phi + \lambda|\beta)(x_\beta|x_{-\beta})_{\mathfrak{d}} && \text{(by (10))} \\
&= (\lambda|\beta)(x_\beta|x_{-\beta}) + u(\lambda, \beta)(x_\beta|x_{-\beta}) && \text{(by (5), (13))} \\
&= 2^{-1}(k_\lambda|(x_\beta|x_{-\beta})k_\beta)_{\mathfrak{d}} + 2^{-1}(k_\lambda|(x_\beta|x_{-\beta})h_\beta)_{\mathfrak{d}} && \text{(by (16), (14))} \\
&= 2^{-1}(k_\lambda|[x_\beta, x_{-\beta}]_{\mathfrak{g}} + [x_\beta, x_{-\beta}]_{\mathfrak{g}'})_{\mathfrak{d}} && \text{(by (4))} \\
&= (k_\lambda|[x_\beta, x_{-\beta}])_{\mathfrak{d}}. && \text{(by (11))}
\end{aligned}$$

(b) $\beta + \gamma \neq 0$: Since $[x_\beta, x_\gamma] \in \mathfrak{n}$, we have that

$$([k_\lambda, x_\beta]|x_\gamma)_{\mathfrak{d}} = (\Phi + \lambda|\beta)(x_\beta|x_\gamma)_{\mathfrak{d}} = 0 = (k_\lambda|[x_\beta, x_\gamma])_{\mathfrak{d}}$$

by (3), (10), (13) and (15).

Case IV. $a = x_\beta$, $b = x_\gamma$, $c = h_\lambda \in \mathfrak{h}$: Since $(\cdot|\cdot)_{\mathfrak{d}}$ is symmetric, we have that

$$([x_\beta, x_\gamma]|h_\lambda)_{\mathfrak{d}} = -(h_\lambda|[x_\gamma, x_\beta])_{\mathfrak{d}} = -([h_\lambda, x_\gamma]|x_\beta)_{\mathfrak{d}} = (x_\beta|[x_\gamma, h_\lambda])_{\mathfrak{d}}$$

by Case II.

Case V. $a = x_\beta$, $b = x_\gamma$, $c = k_\lambda \in \mathfrak{k}$: Since $(\cdot|\cdot)_{\mathfrak{d}}$ is symmetric, we have that

$$([x_\beta, x_\gamma]|k_\lambda)_{\mathfrak{d}} = -(k_\lambda|[x_\gamma, x_\beta])_{\mathfrak{d}} = -([k_\lambda, x_\gamma]|x_\beta)_{\mathfrak{d}} = (x_\beta|[x_\gamma, k_\lambda])_{\mathfrak{d}}$$

by Case III.

Case VI. $a = x_\beta$, $b = h_\lambda \in \mathfrak{h}$, $c = x_\gamma$: Since $(\cdot|\cdot)_{\mathfrak{d}}$ is symmetric, we have that

$$\begin{aligned}
([x_\beta, h_\lambda]|x_\gamma)_{\mathfrak{d}} &= -([h_\lambda, x_\beta]|x_\gamma)_{\mathfrak{d}} = -(h_\lambda|[x_\beta, x_\gamma])_{\mathfrak{d}} \\
&= (h_\lambda|[x_\gamma, x_\beta])_{\mathfrak{d}} = ([h_\lambda, x_\gamma]|x_\beta)_{\mathfrak{d}} = (x_\beta|[h_\lambda, x_\gamma])_{\mathfrak{d}}
\end{aligned}$$

by Case II.

Case VII. $a = x_\beta$, $b = k_\lambda \in \mathfrak{k}$, $c = x_\gamma$: Since $(\cdot|\cdot)_{\mathfrak{d}}$ is symmetric, we have that

$$\begin{aligned}
([x_\beta, k_\lambda]|x_\gamma)_{\mathfrak{d}} &= -([k_\lambda, x_\beta]|x_\gamma)_{\mathfrak{d}} = -(k_\lambda|[x_\beta, x_\gamma])_{\mathfrak{d}} \\
&= (k_\lambda|[x_\gamma, x_\beta])_{\mathfrak{d}} = ([k_\lambda, x_\gamma]|x_\beta)_{\mathfrak{d}} = (x_\beta|[k_\lambda, x_\gamma])_{\mathfrak{d}}
\end{aligned}$$

by Case III.

Case VIII. $a \in \mathfrak{h} \oplus \mathfrak{k}$, $b \in \mathfrak{h} \oplus \mathfrak{k}$, $c = x_\gamma$: Clearly we have that

$$([a, b]|x_\gamma)_\mathfrak{d} = 0 = (a|[b, x_\gamma])_\mathfrak{d}$$

by (8), (12) and (15).

Case IX: $a \in \mathfrak{h} \oplus \mathfrak{k}$, $b = x_\gamma$, $c \in \mathfrak{h} \oplus \mathfrak{k}$: Clearly we have that

$$([a, x_\gamma]|c)_\mathfrak{d} = 0 = (a|[x_\gamma, c])_\mathfrak{d}$$

by (12), (13) and (15).

Case X. $a = x_\gamma$, $b \in \mathfrak{h} \oplus \mathfrak{k}$, $c \in \mathfrak{h} \oplus \mathfrak{k}$: Clearly we have that

$$([x_\gamma, b]|c)_\mathfrak{d} = 0 = (x_\gamma|[b, c])_\mathfrak{d}$$

by (13), (15) and (8).

Case XI. $a \in \mathfrak{h} \oplus \mathfrak{k}$, $b \in \mathfrak{h} \oplus \mathfrak{k}$, $c \in \mathfrak{h} \oplus \mathfrak{k}$: Clearly we have that

$$([a, b]|c)_\mathfrak{d} = 0 = (a|[b, c])_\mathfrak{d}$$

by (8). □

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